INTRODUCTION TO POSITIVE REPRESENTATIONS AND FOCK-GONCHAROV COORDINATES

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Abstract. In this notes, we will try to give a simple description of the set of positive representations of the fundamental group of a surface with non-empty boundary to the group $\mathrm{PSL}_m(\mathbb{R})$, as defined by Fock and Goncharov [1]. The construction uses a special set of coordinates on the space of all representations in $\mathrm{PGL}_m(\mathbb{C})$ now called the Fock-Goncharov coordinates. For each step of the construction, we will consider the classical cases $m = 2$ and $m = 3$ before turning to the general case. We will also give the main property of these representations, namely their faithfulness and discreteness.

1. Introduction

Given a surface $\Sigma_{g,s}$ of genus g with s boundary components, there are well-known coordinates on the (classical) Teichmüller space $\mathbb{T}(\Sigma)$ which are given by Thurston *shearing coordinates*. On the other hand, points in Teichmüller spaces can also be seen as conjugacy classes of representations of the fundamental group of the surface into the group $\mathrm{PSL}_2(\mathbb{R})$ satisfying various properties such as discreteness and faithfulness. In this note, we will explain how to define complex coordinates on the bigger moduli space of representations into $\mathrm{PGL}_2(\mathbb{C})$ (in fact, on a finite ramified cover over this space) and see that Teichmüller space will correspond exactly to the subset of the moduli space for which the coordinates are real positive. What is more interesting is that it is possible to generalize this setting to the case of representations into $\mathrm{PGL}_m(\mathbb{C})$.

The purpose of this note will be to define the space of so-called *framed representations* of a surface group into $\mathrm{PGL}_m(\mathbb{C})$ and define global coordinates on this space using a triangulation of the surface. The coordinates will depend on the triangulation, but the set of representations with positive coordinates will be well-defined and will not depend on the chosen triangulation. This set of *positive representations* will be the so-called *higher Teichmüller space* in the sense of Fock and Goncharov. While the positive representations might be defined for a general split semi-simple real Lie group, using Lustzig notion of total positivity, we will restrict ourselves to the simpler case $G = \mathrm{PSL}_m(\mathbb{R})$ where everything can be made explicit.

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To understand the properties of positive representations, we will see how to construct a representations from a given set of coordinates. We will also see that a positive representation defines a positive map from the Farey set of the surface to the flag variety, and that this notion can be generalized to closed surfaces.

These notes aim to constitute an introduction to the basic ideas, definitions and properties of positive representations. We avoid technical issues in the definitions and statements of theorem and we try to give proofs in an elementary way, when possible. For the complete definitions in full generalities, and detailed proofs, one should obviously refer to the original articles of Fock and Goncharov [1, 2, 3].

2. Framed representations

Let Σ be a compact orientable surface of genus g with $s \geq 1$ open discs removed. The standard presentation of the fundamental group is given by :

$$\pi_1(\Sigma) = \langle A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_s \mid \prod_{i=1}^{g}[A_i, B_i] \prod_{j=1}^{s} C_j \rangle$$

where the C_j correspond to the homotopy type of a curve going aroung the j-th hole. Let G be a Lie group. The moduli space of representations into G is

$$\mathcal{R}_G(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

where the action of G is by conjugation. Note that one should take the GIT quotient instead of the topological quotient to make this space an algebraic variety, but we will avoid this technical discussion in this exposition. For simplicity, one can consider equivalently the usual topological quotient of the subset of completely reducible representations.

We are going to describe a ramified cover of the space $\mathcal{R}_G(\Sigma)$ which will be the space of *framed representations* and will be denoted $\mathfrak{X}_G(\Sigma)$. The additional data is given by choices of flags in $\mathbb{C}^m$, and hence we first recall the necessary results on flags in a vector space, and prove some results on configuration of flags.

2.1. Flags.

Definition 2.1. *A (complete) flag F in a finite dimensional real or complex vector space V is an increasing sequence of subspaces of V such that*

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_m = V$$

with $\dim F_k = k$.

We denote the set of all flags by $\mathrm{Flag}(\mathbb{R}^m)$, $\mathrm{Flag}(\mathbb{C}^m)$, *and when there is no ambiguity on the ambiant field* $\mathcal{F}_n$.

A basis $(e_1, \dots, e_m) \in V^n$ is said to be adapted to the flag if the first k elements provide a basis of F_k. Reciprocally, given a basis of V, the standard flag for the basis is the one where the i-th subspace is spanned by the first i vectors of the basis. The pair of opposite flags associated to the basis $\mathcal{B} = (e_1, \dots, e_m)$ is the pair (F, F') where, F is the standard flag associated to $\mathcal{B}$ and F' is the standard flag for the reversed basis $(e_m, \dots, e_1)$.

The group $\mathrm{GL}_m(\mathbb{C})$ and hence $\mathrm{PGL}_m(\mathbb{C})$ acts naturally on $\mathcal{F}_n$ by left multiplication. This action is naturally transitive as is the action of $\mathrm{PGL}_m(\mathbb{C})$ on the set of basis of V. The stabilizer subgroup $\mathrm{Stab}(F) \subset G$ of a complete flag is identified with the set of invertible upper triangular matrices with respect to any basis adapted to the flag. For convenience, we will denote :

$$B = \left\{ \begin{pmatrix} * & \cdots & & \cdots & * \\ 0 & * & & & \vdots \\ \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & * \end{pmatrix} \right\} \qquad H = \left\{ \begin{pmatrix} * & 0 & \cdots & \cdots & 0 \\ 0 & * & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & * \end{pmatrix} \right\}$$

The subgroup of upper triangular matrices and the maximal torus of $\mathrm{PGL}_m(\mathbb{C})$ with respect to the canonical basis of $\mathbb{C}^m$.

Another point of view is to see an element X of $\mathrm{GL}_m(\mathbb{C})$ as its m column vectors $x^1, \ldots, x^m$. This provides a basis and we can associate the standard flag F^X with respect to this basis. This construction is well-defined for an element of $\mathrm{PGL}_m(\mathbb{C})$, so we get a map $\mathrm{PGL}_m(\mathbb{C}) \to \mathcal{F}_n$. The subgroup B of upper triangular matrices acts from the right on $\mathrm{PGL}_m(\mathbb{C})$, and this action leaves the map invariant. So we have an induced map $\mathrm{PGL}_m(\mathbb{C})/B \to \mathcal{F}_n$. This map is easily showed to be bijective and equivariant with respect to the $\mathrm{PGL}_m(\mathbb{C})$ action on the left. Thus we can identify $\mathcal{F}_n$ with $\mathrm{PGL}_m(\mathbb{C})/B$.

Remark 2.2. *This construction can be generalized in the more general setting of a semi-simple Lie group G. In this case, the group of upper triangular matrices B is replaced with any Borel subgroup, and the quotient G/B is called the flag variety. It parametrizes the set of all Borel subgroups of G*

Let $F = (F_1, \ldots, F_m)$ and $F' = (F'_1, \ldots, F'_m)$ be two complete flags. We say that the pair of flags is in *generic position* if for any k we have $F_k \cap F'_{m-k} = \{0\}$. We say that a k-uple of flag is generic, if any pair of flag is in generic position.

Note that for a generic k-uple of flags $(A^{(1)}, \ldots, A^{(k)})$ in $\mathbb{C}^m$ and dimensions of subspaces $i_1, \ldots, i_k$, the dimension of a direct sum is given by

$$\dim(A^{(1)}_{i_1} + \cdots + A^{(k)}_{i_k}) = \min(m, i_1 + \cdots + i_k)$$

Hence if $i_1 + \cdots + i_k \leq m$ then the sum is direct.

The group $G = \mathrm{PGL}_m(\mathbb{C})$ acts naturally on k-uples of flags by left multiplication. The space of G-orbits of k-uples of flags is called the space of configuration of k flags in $\mathbb{C}^m$, also denoted $\mathrm{Conf}_k(\mathbb{C}^m)$. If we restrict to k-uples of flags in generic position, the space is denoted $\mathrm{Conf}^*_k(\mathbb{C}^m)$.

Lemma 2.3. *The group G acts transitively on the set of couple of flags in generic position. In other words, $\mathrm{Conf}^*_2(\mathbb{C}^m)$ consists of a single point. Moreover, the stabilizer in G of a couple of flag is isomorphic to H.*

Proof. We construct a basis $(e_1, \dots, e_n)$ such that the pair (F, F') is a pair of opposite flags for this basis. It suffices to define e_k as a direction vector of $F_k \cap F'_{m+1-k}$.

In this basis, an element $g \in \mathrm{PGL}_m(\mathbb{C})$ will leave the two flags invariant if and only if it is both upper triangular and lower triangular, hence diagonal. Hence, the stabilizer of the couple of flags will be the set of diagonal matrices in this basis. Note that this stabilizer forms a maximal torus in $\mathrm{PGL}_m(\mathbb{C})$ □

We can be even more precise with this statement

Corollary 2.4. *If A, B, C are three flags in generic position, there exists a basis $\{e_1, \dots, e_m\}$ such that (A, B) is an opposite pair for the basis and the line C_1 is spanned by the vector $e_1 - e_2 + \cdots + (-1)^{m+1} e_m$. Such a basis is unique up to a multiplication by a scalar.*

Using this we can define a map on configuration of four flags. Let $[A, B, C, D] \in \mathrm{Conf}_4(\mathbb{C}^m)$. There exists a basis $\mathcal{B} = (e_1, \dots, e_m)$ (resp. $\mathcal{B}'$, unique up to multiplication, such that A and C are the standard and opposite flags and the subspace B_1 (resp. D_1) is spanned by $e_1 - \cdots + (-1)^m e_m$. The transition matrix from $\mathcal{B}$ to $\mathcal{B}'$ is a well-defined element of $\mathrm{PGL}_m(\mathbb{C})$ which is in the stabilizer of the two flags A and C and hence can be identified with an element of H. Hence we get a map $p : \mathrm{Conf}_4(\mathbb{C}^m) \to H$. Using this we can state the following

Lemma 2.5. *The map*

$$\begin{aligned} \mathrm{Conf}_4(\mathbb{C}^m) &\longrightarrow \mathrm{Conf}_3(\mathbb{C}^m) \times H \times \mathrm{Conf}_3(\mathbb{C}^m) \\ [A, B, C, D] &\longmapsto ([A, B, C], p([A, B, C, D]), [A, C, D]) \end{aligned}$$

is a birational isomorphism.

Remark 2.6. *We won't expand on the notion of birational isomorphism. The main idea is that both sides can be considered as algebraic varieties and that they are isomorphic outside some lower dimensional subsets. Here these lower dimensional subsets will correspond to non-generic configurations of flags.*

2.2. **Ciliated surfaces and framed representations.**

We note $\hat{\Sigma}$ the surface Σ together with a set of marked points P on the boundary components of Σ defined up to an isotopy. The j-th boundary component of Σ should have p_j marked points where $p_j \geq 0$. A boundary component with no marked point ($p_j = 0$) will be called a hole. The boundary $\partial\hat{\Sigma} = \partial\Sigma \setminus P$ is a union of circles (holes) and arcs (cilium). The pair $\hat{\Sigma} = (\Sigma, P)$ is called a ciliated surface.

Definition 2.7. *A framed representation of $\hat{\Sigma}$ is given by the data of a representation $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}_m(\mathbb{C}))$ and flags $(F^{(1)}, \dots, F^{(t)})$ in $\mathrm{Flag}(\mathbb{C}^m)$ associated to each connected component of $\partial\hat{\Sigma}$, such that if $C_j \in \pi_1(\Sigma)$ corresponds to a boundary curve with no marked point (a hole), then the corresponding flag is invariant by $\rho(C_j)$.*

The group $G = \mathrm{PGL}_m(\mathbb{C})$ acts naturally on framed representations by conjugation on the representation and left multiplication on the flags. Hence we define the moduli space of framed representation as the quotient of the set of framed representations by the action of G, and we denote it by $\mathfrak{X}_G(\hat{\Sigma})$.

Examples :

- A fundamental example, if Σ is a disc, then $\pi_1(\Sigma)$ is trivial and $\hat{\Sigma}$ is a disc with k marked points on the boundary. In this case $\mathfrak{X}_G(\hat{\Sigma})$ is exactly $\mathrm{Conf}_k(\mathbb{C}^m)$, the space of configuration of k flags in $\mathbb{C}^m$.
- In the case when Σ has no marked point, we can identify Σ with $\hat{\Sigma}$. Hence $\partial\hat{\Sigma}$ is a union of circles and all the flags associated to a boundary curve must be invariant by the corresponding holonomy along this curve.

When $\Sigma = \hat{\Sigma}$, a generic element $\rho(C_j) \in \mathrm{PGL}_m(\mathbb{C})$ corresponding to a hole have m invariant eigendirections. Hence there are $m!$ choices of invariant flags over a hole corresponding to all possible ordering of the projective basis of eigendirections. However, when the eigenvalues are not all distinct (for example when the element is parabolic), then there are fewer choice of invariant flags, (and possibly only one). This shows that the space $\mathfrak{X}_{\mathrm{PGL}_m(\mathbb{C})}(\Sigma)$ is a ramified cover over $\mathbb{R}_{\mathrm{PGL}_m(\mathbb{C})}(\Sigma)$ of degree $n!$.

2.3. **Triangulations of surfaces.** A *triangulation* of a ciliated surface $\hat{\Sigma} = (\Sigma, P)$ is a maximal collection of disjoint simple non-homotopic arcs joining boundary components in $\partial\hat{\Sigma}$. An *external arc* is an arc retractible on the boundary and joining two adjacent cilium. Internal arcs are those not of this type.

A triangulation decomposes the surface into triangles. An external (resp. internal) arc will be an arc contained in one (resp. two) triangles. A triangulation is equivalently given by an ideal triangulation on the punctured surface obtained by shrinking holes to points, and adding a puncture on each boundary segment which is not a hole (in some sense forgetting the marked points by taking a dual set of points on each boundary component).

Let T be a triangulation, and $V(T), E(T), F(T)$ the set of vertices, edges and faces of the triangulation T. Note also $E_e(T)$ and $E_i(T)$ the set of external edges and internal edges respectively.

Topologically, a ciliated surface $\hat{\Sigma}$ is defined by its genus and the finite collection $P = (p_1, \ldots, p_s)$ of number of marked points on the j-th boundary component. Denote the number of holes by h and the number of cilia by c. Note that $c = \sum p_i$. The number of faces, vertices and edges of a triangulation of the surface are determined by the topology of $\hat{\Sigma}$ and we have :

(1) $|V(T)| = h + c$
(2) $|E_e(T)| = c$
(3) $|E_i(T)| = |E(T)| - |E_0(T)| = 6g - 6 + 3s + c.$
(4) $|E(T)| = 6g - 6 + 3s - 2c$
(5) $|F(T)| = 4g - 4 + 2s - c$

2.4. **Decomposition Theorem.** The first step to describe coordinates for the spaces $\mathfrak{X}_G(\hat{\Sigma})$ is to start from a triangulation T of the surface. The coordinates on $X_G(\hat{\Sigma})$ will be given by coordinates on the space of triple of flags and also by coordinates on the space of possible gluings of two triples of flags along one edge.

Hence, the coordinates are divided into two groups :

- The triangle invariants which correspond to coordinates on $\mathrm{Conf}_3^*(\mathbb{C}^m)$.
- The edge invariants which parametrize the possible gluing of two triple of flags along an edge. As we showed previously, this is parametrized by H.

We get the following result :

Theorem 2.8. *Let $\hat{\Sigma}$ be a ciliated surface and T a triangulation. There is a birrational isomorphism :*

$$\pi_T : \mathfrak{X}_G(\hat{\Sigma}) \longrightarrow \prod_{t\in F(T)} (\mathrm{Conf}_3^*(\mathbb{C}^m)) \times \prod_{e\in E_i(T)} H.$$

Proof. We give a sketch of the proof here and refer to [1] for more details.

First we can prove easily the theorem when $\hat{\Sigma} = \hat{D}_k$ by induction on the number of points using the previous lemmas. Indeed, a configuration of $k+1$ points can be considered as the union of a configuration of k points and a configuration of three points such that we glue an exterior edge of each configuration. So we have the decomposition theorem for the configuration of k flags, which is exactly the space of framed representations in this case, as the fundamental group of a disc is trivial.

For a general surface $\hat{\Sigma}$, we first need to define the map π_T. To do that we lift the triangulation T to the universal cover $\widetilde{\Sigma}$. This gives an identification of the triangulated universal cover with the hyperbolic plane triangulated by the Farey triangulation. We can then construct a map f_ρ from the set of vertices of the lifted triangulation into $\mathcal{F}_m$ in a ρ-equivariant way. This map is uniquely defined up to conjugacy. Now, we can choose a fundamental polygonal domain for Σ in $\widetilde{\Sigma}$ as a union of triangles. This gives a configuration of flags, and hence we can use the result on $\hat{D}_k$ to get our map. Note that by ρ-equivariance, this does not depend on the choice of a fundamental domain.

Now to prove the theorem, we need to show that we can reconstruct both the framing and the holonomy representation from the data. To do that we lift the triangulation T to the universal cover $\widetilde{\Sigma}$. This gives an identification of the triangulated universal cover with the hyperbolic plane triangulated by the Farey triangulation. This gives naturally a framing explicitly given by the configuration of flags in the data. To recover the holonomy representation consider any path $\gamma \in \pi_1(\Sigma)$. We can find a finite polygon in the universal cover containing the lift of the path. This lift start in a specific triangle Δ and end in the triangle $\gamma \cdot \Delta$. These two triangles define two triple of flags in the same configuration. The unique (up to conjugacy) element sending the starting triple of flag to the ending triple of flag corresponds to $\rho(\gamma)$. It does not depends on the choices of a lift or a polygon. Moreover, if γ is homotopic to a boundary component, then the triangles that are crossed by the lift of the path all share one vertex in $\hat{\Sigma}$. This means that the element ρ_γ here defined, will preserve the flag corresponding to this vertex. Hence, this defines a framed representation of $\pi_1(\Sigma)$.

$\square$

3. Coordinates on moduli space of framed structure

This section is devoted to the construction of explicit complex coordinates on the space of configurations of three and four flags, which will then be used to define coordinates on the space of framed representations using the decomposition theorem. The main feature of these coordinates is that they produce a positive structure on the space $\mathfrak{X}_G(\hat{\Sigma})$ in the sense that the set of points which have positive coordinates is well-defined (and does not depend on the triangulation). The so-called higher Teichmüller space will be exactly the set of points with positive coordinates.

One of the important result of Fock and Goncharov is the construction of a positive structure on $\mathfrak{X}_G(\hat{\Sigma})$ for any split semi-simple real Lie group G, which allows them to define higher Teichmüller spaces in a very broad context. However, explicit coordinates can only be given in the $\mathrm{PGL}_m(\mathbb{C})$-case, which is the focus of this note.

We will first study the classical case of $\mathrm{PGL}_2(\mathbb{C})$ where the invariants will be related to Thurston shear coordinates on Teichmüller space. Then, we will see the case of $\mathrm{PGL}(3,\mathbb{C})$ where one has to understand the triple-ratio of a triple of flags in $\mathbb{C}^3$. The general coordinates will easily be constructed from this two basic cases.

3.1. The $m = 2$ case.

3.1.1. *Triples of flags.* We first describe the space $(\mathrm{Conf}_3^*(\mathbb{C}^2))$ of configurations of three flags. A flag in $\mathbb{C}^2$ is simply given by a direction in $\mathbb{C}^2$ or equivalently a point in $\mathbb{C}P^1$. The group $\mathrm{PGL}_2(\mathbb{C})$ is the group of Möbius transformation of $\mathbb{C}P^1$ whose action is simply 3-transitive on $\mathbb{C}P^1$. In other words, $\mathrm{PGL}_2(\mathbb{C})$ acts transitively on generic triples of flags in $\mathbb{C}P^1$. So there is only one orbit of generic triples of flags. Hence, the space $\mathrm{Conf}_3^*(\mathbb{C}^2)$ is reduced to a single point.

3.1.2. *Quadruple of flags and cross-ratios.* Now, let A, B, C, D be four flags in $\mathbb{C}^2$ in generic position, assimilated to points in $\mathbb{C}P^1$. We can consider that these four flags are attached to the vertices of a quadrilateral. Take the triangulation of this quadrilateral with an edge, denoted AC, joining (the vertices associated to) A and C. We are going to associate a coordinate to this configuration and attach it to the edge AC.

We will express explicitly the projection on H from our previous construction in this simple case. Take the basis $\mathcal{B} = (e_1, e_2)$ such that A_1 is spanned by e_1, C_1 is spanned by e_2 and B_1 is spanned by $e_1 - e_2$. And similarly take the basis $\mathcal{B}'$ such that D_1 is spanned by $e'_1 - e'_2$. So we get $e'_1 = xe_1$ and $e'_2 = ye_2$ for some $x, y \in \mathbb{C}$. Without loss of generality we can choose $xy = 1$ and hence we get the element $g = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in H$ seen as an element of $\mathrm{PGL}_2(\mathbb{C})$. Now a coordinate on H is given by the map $g \mapsto x^2$.

This definition is equivalent to the following one : As the action on triple of points in $\mathbb{C}P^1$ is simply transitive, we can send the first triangle (A, B, C) to $(0, -1, \infty)$ and then the fourth vertex is sent to a well-defined point $x \in \mathbb{C}P^1$.

In fact, the coordinate of the point x is given by the cross ratio of the 4 points (A, B, C, D) (in the correct order) in $\mathbb{C}P^1$ denoted by :

$$[A, B, C, D] = \frac{(A - D)(B - C)}{(A - C)(B - D)}$$

Note that the sign convention that we adopt might be different from the usual definition of the cross-ratio. This is due to the fact that we want the cross-ratio to be positive when the two triangles defined by the diagonal of the quadrilateral are disjoint (and hence talk about positive representations instead of negative ones).

3.1.3. *General Surface.* Now let Σ be a general surface. Let $[\hat{\rho}]$ be a conjugacy class of framed representation and T a triangulation of Σ. From the decomposition theorem and the previous discussion on configuration of three and four flags, we get the following birational isomorphism :

$$\Phi_T : \mathfrak{X}_{\mathrm{PGL}_2(\mathbb{C})}(\Sigma) \longrightarrow (\mathbb{C}^*)^{|E_i(T)|}$$

This map gives a set of coordinates on the moduli space of framed representations.

Note that when one restricts to fuchsian representations, these coordinates correspond exactly to Thurston shear coordinates.

3.2. The $m = 3$ case.

3.2.1. *Triple ratios.* We need to understand the moduli space of configuration of triples of flags in $\mathbb{C}^3$, and find coordinates on it. A flag in $\mathbb{C}^3$ is given by a line and a plane, or equivalently by a point and a line in $\mathbb{C}P^2$. Heuristically, the space of triples of flags is of dimension 9 (dimension 3 for each flag) while the group $G = \mathrm{PGL}_m(\mathbb{C})$ is of dimension 8. Hence the moduli space of configuration should be of dimension 1, and we shall need only one invariant to describe it. This invariant is given by the triple ratio defined as follows.

Let $A = (A_1, A_2)$, $B = (B_1, B_2)$ and $C = (C_1, C_2)$ be a generic triple of flag, where $\dim A_i = i$. Choose v_A, v_B, v_C some direction vectors for the lines A_1, B_1 and C_1. and let $f_a, f_b, f_c \in (\mathbb{C}^3)^*$ be linear forms representatives defining the planes A_2, B_2 and C_2. The triple ratio is given by :

$$X := r_3(A, B, C) = \frac{f_a(v_B) f_b(v_C) f_c(v_A)}{f_a(v_C) f_b(v_A) f_c(v_B)}$$

This does not depend on the choices of the direction vectors or the linear forms, and is invariant under the action of $\mathrm{PGL}_m(\mathbb{C})$ on triple of flags. This quantity is a complete invariant of generic configuration of three flags. Namely, we get the following lemma :

Lemma 3.1. *Let A, B, C and A', B', C' be two triples of flags such that $r_3(A, B, C) = r_3(A', B', C')$. Then there exists a unique element $g \in \mathrm{PGL}_m(\mathbb{C})$ such that $g(A) = A'$, $g(B) = B'$ and $g(C) = C'$.*

Proof. From Corollary **??** we know that there exists a unique element $g \in \mathrm{PGL}_m(\mathbb{C})$ such that $g(A') = A$, $g(B') = B$ and $g(C_1') = C_1$, so we can consider the triple $(A, B, g(C'))$. The equality of triple ratios proves that $\frac{f_c(v_A)}{f_c(v_B)} = \frac{f_{c'}(v_A)}{f_{c'}(v_B)}$. And as we also know that $f_c(v_C) = f_{c'}(v_C) = 0$ we have that f_c and $f_{c'}$ are equal up to a multiplicative constant and hence they define the same plane. Hence $g(C') = C$ and the lemma is proved. $\square$

We can now state the following proposition :

Proposition 3.2. *The map*

$$\begin{aligned} \mathrm{Conf}_3^*(\mathbb{C}^3) &\longrightarrow \mathbb{C}^* \\ [(A, B, C)] &\longmapsto r_3(A, B, C) \end{aligned}$$

is a birational isomorphism.

3.2.2. *Quadruple of flags.* Now we can look at the moduli space of configuration of four flags in $\mathbb{C}^3$. Let A, B, C, D be four flags in $\mathbb{C}^3$ in generic position and such that we have $r_3(A, B, C) = X$ and $r_3(A, C, D) = Y$. We want to construct invariants using cross-ratio of quadruples of points in $\mathbb{C}P^1$ associated to the edge AC. Note that four planes in $\mathbb{C}^3$, sharing a line in common, define four lines in $\mathbb{C}^2$ by projection, and hence can be associated to a quadruple of points in $\mathbb{C}P^1$.

The four planes $(A_2, A_1 \oplus B_1, A_1 \oplus C_1, A_1 \oplus D_1)$ all contain the line A. Hence, we can define their cross-ratio Z and get an element in $\mathbb{C}^*$. Similarly the four planes $(C_2, C_1 \oplus D_1, C_1 \oplus A_1, C_1 \oplus B_1)$ all contain the line C_1. So we can also define their cross-ratio W. From this we get :

Proposition 3.3. *The map*

$$\begin{aligned} \mathrm{Conf}_4^*(\mathbb{C}^3) &\longrightarrow (\mathbb{C}^*)^4 \\ [(A, B, C, D)] &\longmapsto (X, Y, Z, W) \end{aligned}$$

is a birational isomorphism.

3.2.3. *General Surface.* Now let Σ be a general surface. Let $[\hat{\rho}]$ be a conjugacy class of framed representation and T an ideal triangulation of Σ. From the decomposition theorem and the previous discussion on configuration of three and four flags, we get the following birational isomorphism :

$$\Phi_T : \mathfrak{X}_{\mathrm{PGL}_3(\mathbb{C})}(\hat{\Sigma}) \longrightarrow (\mathbb{C}^*)^{2|E_0(T)|+|F(T)|}$$

which gives $16g - 16 + 8s$ coordinates for the moduli space of framed representations. More details on this case can be found in the Fock-Goncharov article [3].

3.3. **Coordinates in the general case.** When $m \geq 3$ the same ideas are used to construct coordinates. The triangle invariants are computed by restricting the flags in $\mathbb{C}^m$ to $\mathbb{C}^3$ and the edge invariants are computed by finding four planes sharing a line in $\mathbb{C}^3$.

3.3.1. *Invariants of triples of flags.* Let $m \geq 3$ and A, B and C be a generic triple of flags in $\mathbb{C}^m$. Denote by A_k the subspace of dimension k in the flag A. The main idea to get invariants of triple of flags is to extract from the three flags in $\mathbb{C}^m$, a certain number of triples of flag in $\mathbb{C}^3$ in a clever way.

Let $i, j, k \geq 1$ such that $i + j + k = m$. The direct sum $W_{i,j,k} = A_{i-1} \oplus B_{j-1} \oplus C_{k-1}$ is a subspace of dimension $m - 3$. Hence the quotient $V_{i,j,k} = \mathbb{C}^m / W$ is isomorphic to $\mathbb{C}^3$.

We can take the projection of the subspace A_i and A_{i+1} on $V_{i,j,k}$, and this give a flag $A^{(i,j,k)} = (\overline{A}_i, \overline{A}_{i+1})$ in $V_{i,j,k}$. Similarly, we get two other flags $B^{(i,j,k)}$ and $C^{(i,j,k)}$ by projecting B and C on $V_{i,j,k}$. Hence we have a triple of flags in $\mathbb{C}^3$ and hence we can compute :

$$X_{i,j,k}(A, B, C) = r_3(A^{(i,j,k)}, B^{(i,j,k)}, C^{(i,j,k)})$$

Obviously, this quantity is an invariant of triples of flags in $\mathbb{C}^m$. For each $i + j + k = m$ we get another invariant. This gives $\frac{(m-1)(m-2)}{2}$ coordinates in $\mathbb{C}^*$. We see that this number should be equal to the dimension of the space of configuration of three flags in $\mathbb{C}^m$. Indeed we have

Proposition 3.4. *The map*

$$\begin{aligned} \mathrm{Conf}_3(\mathbb{C}^m) &\longrightarrow (\mathbb{C}^*)^{\frac{(m-1)(m-2)}{2}} \\ (A, B, C) &\longmapsto \{X_{i,j,k}(A, B, C) | i, j, k \geq 0, \quad \textit{and } i + j + k = m, \} \end{aligned}$$

is a birational isomorphism.

3.3.2. *Invariants of quadruples of flags.* Let A, B, C, D be a quadruple of flags in $\mathbb{C}^m$. The main idea is to extract several quadruples of lines in $\mathbb{C}^2$, that we will associate to this edge

Let $i, j \geq 1$ such that $i + j = m$. The direct sum $U_{i,j} = A_{i-1} \oplus C_{j-1}$ is a subspace of dimension $m - 2$. Hence the quotient $V_{i,j} = \mathbb{C}^m / U_{i,j}$ is isomorphic to $\mathbb{C}^2$

We can take the projection of A_i, B_1, C_j and D_1 that we denote $A^{(i,j)}$, $B^{(i,j)}$, $C^{(i,j)}$ and $D^{(i,j)}$ on this subspace. And this will give a quadruple of lines in $\mathbb{C}^2$ so we can compute :

$$X_{i,j}(A, B, C, D) = [A^{(i,j)}, B^{(i,j)}, C^{(i,j)}, D^{(i,j)}]$$

This quantity is an invariant of quadruple of flags in $\mathbb{C}^m$. For each $i + j = m$ we get another invariant. This gives $m - 1$ coordinates in $\mathbb{C}^*$ that we attach to an edge. This is exactly the dimension of a maximal torus H in $\mathrm{PGL}_m(\mathbb{C})$.

3.3.3. *Coordinates for a general surface.* Let Σ be a general surface, with a triangulation. To visualize easily all the invariants and their relation, we use an m-triangulation of the surface Σ, where each original triangle is divided into more triangles as seen on the figure below. Each vertex of this m-triangulation correspond to a triple (i, j, k) such that $i, j, k \geq 0$ are integers and $i + j + k = m$.

For each point, except the vertices of the original triangle, on this m-triangulation, we associate a coordinate. Namely, to each vertex in the interior of a triangle corresponds a triple $i, j, k \geq 1$ such that $i + j + k = m$ so we can associate a triangle invariant. and for

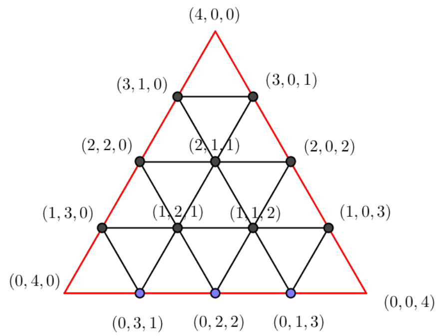

FIGURE 1. 4-triangulation : Each point corresponds to a triple (i, j, k) where at least two are non-zero

each vertex in the interior of a side of the triangle corresponds a couple $i, j \geq 1$ such that $i + j = m$ so we can associate an edge invariant.

Let $N(\Sigma, m)$ the number of vertices of this m-triangulation of the surface. We get

$$N(\Sigma, m) = |F(T)| \left(\frac{(m-1)(m-2)}{2} \right) + |E(T)|(m-1)$$

So with the same principle as before, this gives a map

$$\Phi_T : \mathfrak{X}_{\mathrm{PGL}_m(\mathbb{C})}(\Sigma) \longrightarrow (\mathbb{C}^*)^{N(\Sigma,m)}$$

In the next section, we construct the inverse map Φ_T^{-1}. We start with coordinates in $(\mathbb{C}^*)^{N(\Sigma,m)}$ and we find an element of $\mathfrak{X}_{\mathrm{PGL}_m(\mathbb{C})}(\Sigma)$.

4. Construction of a framed representation from coordinates

4.1. **General Strategy.** Starting from the triangulation, we construct a graph Γ embedded into the surface by drawing long edges transversal to each side of the triangles and inside each triangle connect the ends of edges pairwise by three small edges. Orient the small edges in the triangles in the counterclockwise direction and the long edges crossing the triangulation in an arbitrary way (see Figure 2).

To construct a framed representation, we assign to every oriented edge $e \in \Gamma$, an element in $\mathrm{PGL}_m(\mathbb{C})$ that will depend on the coordinates associated to the corresponding edge or triangle.

For any long edge $e \in \Gamma$, we associate a matrix $E(\{X_{i,j}\})$ which will depend on the edge coordinates $X_{i,j}$ with $i+j = m$ of the corresponding edge of the triangulation. And to any small edge $e \in \Gamma$, and we associate the matrix $T(\{X_{i,j,k}\})$ which will depend on triangle coordinates of the triangle in which it stands.

Given this, for any closed path $\gamma \in \pi_1(\Sigma)$ on the surface, there is a path on Γ homotopic to it. So one can assign to γ an element of $\mathrm{PGL}_m(\mathbb{C})$ by multiplying the group elements (or their inverse if the path goes along the edge against its orientation) assigned to the

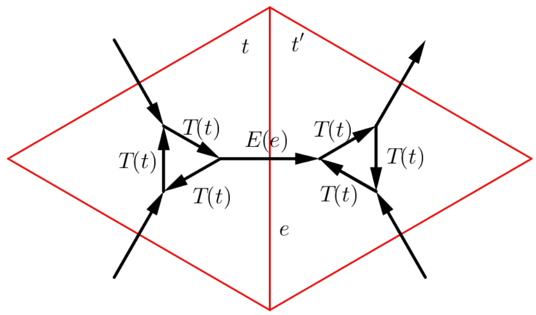

FIGURE 2. The oriented graph dual to the triangulation

edges followed by the path. When the matrices T and E are well constructed, this defines a framed representation of the fundamental group of Σ, with the appropriate coordinates.

The main idea is the following. Let t be a triangle with flags (A, B, C) at each of its vertices and with invariants $X_{i,j,k}$. Then the matrix $T(t)$ associated to this triangle will correspond to the element of $\mathrm{PGL}_m(\mathbb{C})$ that will perform the cyclic permutation of the flags $(A, B, C) \mapsto (B, C, A)$.

Similarly, if (A, B, C, D) is a quadruple of flags with coordinates $X_{i,j}$ associated to the edge (A, C). Then the matrix $E(e)$ will correspond to the element of $\mathrm{PGL}_m(\mathbb{C})$ that sends (A, C) to (C, A) and moreover sends B_1 to D_1.

We will see explicit constructions in the cases $m = 2$ and $m = 3$. The explicit construction in the general case is a little more difficult to compute but follow the same principle.

4.2. **The case $m = 2$.** Let (A, B, C) be a triple of flags in $\mathbb{C}^2$ or equivalently, three points in $\mathbb{C}P^1$. For the matrix T, we take the element sending the triple (A, B, C) to (B, C, A), and we can choose (A, B, C) to be $(0, \infty, -1)$ so that

$$T = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

Now let (A, B, C, D) be a quadruple of points such that the coordinates associated to the edge AC is equal to $z \in \mathbb{C}^*$. For convenience, we may take $(A, B, C, D) = (0, -1, \infty, z)$. Now the matrix $E(z)$ should be the element sending $(0, \infty)$ to $(\infty, 0)$ and moreover sending -1 to z. This is

$$E(z) = \begin{pmatrix} 0 & z \\ -1 & 0 \end{pmatrix}$$

This setting will give the representation of $\pi_1(\Sigma)$ into $\mathrm{PGL}_m(\mathbb{C})$.

To recover the framing over holes, notice that for any coordinate z a product of the form $E(z)T$ is a lower-triangular matrix (while $E(z)T^{-1}$ is upper lower triangular). When following a path around the boundary we always turn right (resp. left), and hence we get only get T (resp. T^{-1}), and hence only one type of product in the formulas. So the monodromy $\rho(C_j)$ is a product of upper triangular (or lower-triangular) matrices, which gives a preferred eigendirection for each boundary, and hence an invariant flag.

4.3. **The case m=3.** Let A, B, C be a triple of flag in $\mathbb{C}^3$, with triple ratio $r_3(A, B, C) = X$. We want to construct an element g that will send (A, B, C) on (B, C, A).

By the transitivity of the G-action on couple of flags, we can assume without loss of generality that we have a basis (e_1, e_2, e_3) such that :

$$A_1 = e_1, \qquad A_2 = e_1 \oplus e_2, \qquad B_1 = e_3, \qquad B_2 = e_3 \oplus e_2$$

and moreover such that

$$C_1 = e_1 - e_2 + e_3$$

In this case, a linear form generating C_2 is given by $Xe_1^* + (1+X)e_2^* + e_3^*$. So the desired matrix in the basis (e_1, e_2, e_3) is given by :

$$T(X) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ X & 1+X & 1 \end{pmatrix}$$

Now let A, B, C, D be four flags in $\mathbb{C}^3$ such that the invariant associated to the edge AC are Z and W. Again, we can assume that we have a basis such that A and C are the standard flags $(e_1, e_1 \oplus e_2)$ and $(e_3, e_3 \oplus e_2)$ respectively, and moreover such that B_1 is generated by $e_1 - e_2 + e_3$. Then, by definition of the two cross-ratios Z and W we have that the vector $Z^{-1}e_1 + e_2 + We_3$ is a direction vector of D_1. In this basis, the element sending (A, C) to (C, A) and B_1 to D_1 is given by

$$E(Z, W) = \begin{pmatrix} 0 & 0 & Z^{-1} \\ 0 & -1 & 0 \\ W & 0 & 0 \end{pmatrix}$$

4.4. **General case.** The general case is obtained by the same principle. Let A, B, C be a triple of flag in $\mathbb{C}^m$ with coordinates given by $X_{i,j,k}(A, B, C)$ with $i, j, k \geq 1$ and $i + j + k = m$. The matrix $T(t) \in \mathrm{PGL}_m(\mathbb{C})$ sending the triple (A, B, C) on (B, C, A) is given by the following formula :

$$T(t) = \left[\prod_{j=m-1}^{1} \left(\left(\prod_{i=j+1}^{m-1} H_i(X_{m-i,i-j,j}) F_i \right) F_m \right) \right] S.$$

where we set $H_i(x) := \mathrm{diag}(\underbrace{x, \ldots, x}_{i}, 1, \ldots, 1)$ and

$$F_i = \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & \\ \vdots & \ddots & 1 & 0 & & \vdots \\ & & 1 & 1 & & \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & & & 0 & 1 \end{pmatrix} \quad , \quad S = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & \iddots & -1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & \iddots & \iddots & & \vdots \\ (-1)^m & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Now let A, B, C, D be four flags with invariants associated to the edge $e = AC$ given by the numbers $x_i = X_{i,m-i}$. The matrix sending (A, C) to (C, A) such that B_1 is sent to D_1 is given by :

$$E(e) = \operatorname{diag}\left((x_1 \cdots x_{m-1}), (x_2 \cdots x_{m-1}), \ldots, x_{m-1}, 1\right) S$$

We won't give the proof of these formulas but the interested reader can find all the details in [1] **9.8**. Note that we get back the matrices in $\mathrm{PGL}_m(\mathbb{C})$ obtained in the $m = 2$ and $m = 3$ case, with these general formulas.

5. Positive Representations

We now have all the tools to define the set of positive representations as a subset of the space of framed representations.

5.1. **Positive part of the moduli space.** The coordinates constructed depend on the chosen triangulation. An important thing is to understand how these coordinates change when we choose a different triangulation of the surface Σ. A classical result states that two triangulation are related by a sequence of flips. Hence we need to understand what happens to the coordinates when we perform a flip along an edge of a triangulation.

In the case $m = 2$. It is an easy exercise to recover the following formulas in Figure 3.

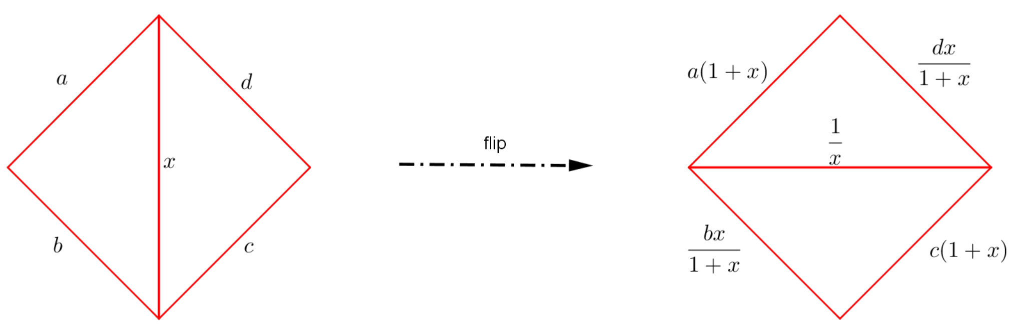

Figure 3. Change of coordinates after a flip

5.1.1. *Cluster mutations.* There is a simple interpretation of these coordinate changes in terms of cluster algebra. From a triangulation of the surface Σ, we can define an oriented graph (a quiver) associated to it (see Fig. 4). Flipping along an arc of the triangulation is equivalent to a performing a mutation of the quiver along a vertex. Such mutation are defined for any quiver as follows :

Let Q be a quiver and k one of its vertex, the mutated quiver $\mu_k(Q)$ is obtained by performing the three following steps :

(1) For each path of length two of the form $i \longrightarrow k \longrightarrow j$, add an arrow $i \longrightarrow j$
(2) Reverse each arrow incident to the vertex k.
(3) Erase any 2-cycles that could have been created by previous steps.

The new coordinates are given by the so-called mutations formulas, which were discovered by Fomin and Zelevinsky in a much broader context, and depend only on the quiver. If $(x_1, \ldots, x_n)$ are the coordinates associated to each vertex of the quiver Q. Let $\varepsilon(i,j)$ be the number of oriented arrows between the vertex i and j. Then the new coordinates $(x'_1, \ldots, x'_n)$ of the quiver $\mu_k(Q)$ are given by

$$x'_j = \begin{cases} \dfrac{1}{x_j} & \text{if } j = k, \\ x_j(1+x_k)^{\varepsilon(j,k)} & \text{if } \varepsilon(j,k) \geq 0, \\ x_j(1+x_k^{-1})^{\varepsilon(j,k)} & \text{if } \varepsilon(j,k) < 0 \end{cases}$$

For a detailed exposition of the relation between cluster algebras and triangulations of hyperbolic surfaces, one should refer to the article [5] of Fomin, Shapiro and Thurston.

5.1.2. *Flips as sequence of mutations.* In the general case, the explicit formulas for the change of coordinates after a flip could be computed directly but would be quite heavy. However, it is again possible to interpret these changes in terms of cluster mutation. We can associate a quiver to any m-triangulation of the surface as indicated in the figure 4. In this case, changing the triangulation is equivalent to a particular sequence of quiver mutations, and the change of coordinates corresponds to the associated cluster mutations (see Chapter **10** of [1]).

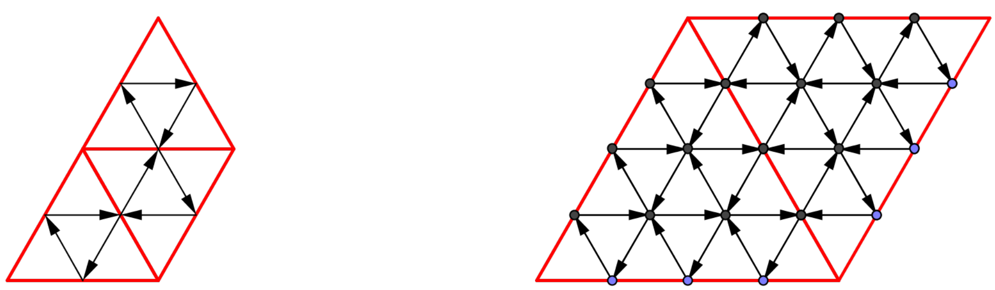

FIGURE 4. Quiver associated to a triangulation and to an m-triangulation of the surface

We notice that the formulas for cluster mutations are subtraction-free. Hence, any change of triangulation gives a subtraction-free change of coordinates. This defines a *positive atlas* on the space of framed structure. Hence, it is clear that if a framed representation has positive coordinates in a given triangulation, then it will have positive coordinates with respect to any triangulation. This allows us to define

Definition 5.1. *A framed representation is said to be* positive *if its coordinates with respect to one (and hence any) triangulation are all real positive.*

The set of positive framed representation is denoted $\mathfrak{X}_G^+(\Sigma)$*.*

A positive representation is just the image of a positive framed representation through the forgetful map $\mathfrak{X}_G(\Sigma) \to \mathcal{R}_G(\Sigma)$*.*

In the next two paragraphs, we will give sketches of proofs of the fundamental properties of positive representations, in particular that they are faithful and discrete.

5.2. **Totally positive matrices and faithfulness of positive representations.**

Theorem 5.2. *Let* $[\rho]$ *be a positive framed representation. Then the corresponding representation* ρ *is faithful. Moreover, for any element* $\gamma \in \pi_1(\Sigma)$ *which is non-trivial and non-boundary, the element* $\rho(\gamma)$ *is positive hyperbolic.*

To prove this theorem, we must look at the construction of the framed structure from the coordinates. Let γ be a non-trivial non-boundary loop. Then the element $\rho(\gamma)$ is of the form

$$\rho(\gamma) = E(e_1)(T(t_1))^{\epsilon_1} E(e_2)(T(t_2))^{\epsilon_2} \cdots E(e_p)(T(t_p))^{\epsilon_p}$$

Suppose all the coordinates are positive. Then we notice that any matrix of the form $E(e)T(t)$ (resp. $E(e)(T(t))^{-1}$) will be a totally positive upper-triangular matrix (resp. totally positive lower triangular matrix). The product of totally positive matrices is a totally positive matrix. And totally positive matrices have the property to have real, positive and distinct eigenvalues. Hence, $\rho(\gamma)$ is positive hyperbolic. The faithfulness is easily deduced from that as a positive hyperbolic element is different from the identity.

5.3. **Farey set and discreteness.** To prove discreteness of positive representation, we need to define a new object. So let $\hat{\Sigma}$ be a ciliated surface and T a triangulation.

Definition 5.3. *The Farey set of the surface* $\hat{\Sigma}$*, denoted* $\mathcal{F}_\infty(\hat{\Sigma})$ *is a cyclic* $\pi_1(\Sigma)$*-set defined as follows :*

shrink the holes of Σ *to punctures and lift to the universal cover, which is an open disc. The set* $\mathcal{F}_\infty(\Sigma)$ *is the set of lifts of punctures which is a countable subset on the boundary of the disc. The group* $\pi_1(\Sigma)$ *acts naturally by deck transformation, and the cyclic structure is given by the cyclic structure on the boundary of the disc.*

Starting from a framed representation $[\rho]$, one can define a ρ-equivariant map :

$$\beta_\rho : \mathcal{F}_\infty(\Sigma) \longrightarrow \mathrm{Flag}(\mathbb{R}^m)$$

which is well-defined modulo conjugation by $\mathrm{PGL}_m(\mathbb{C})$.

We can defined framed representations using only maps from the Farey set to the flag variety as follows as there is a natural bijection between $\mathfrak{X}_{\mathrm{PGL}_m(\mathbb{C})}(\hat{\Sigma})$ and the set of ρ-equivariant maps $\mathcal{F}_\infty(\Sigma) \longrightarrow \mathcal{F}_m$.

Definition 5.4. *A triplet of flags (A, B, C) is* positive *if it is equivalent (modulo the action of $\mathrm{PGL}_m(\mathbb{C})$) to a triple $(F^+, F^-, u \cdot F^-)$ where F^+ and F^- are the standard flag and u is a totally positive upper triangular matrix.*

A configuration of flag $(F_1, \ldots, F_n)$ is positive if any oriented triplet of flag is positive.

If $[\rho]$ is a positive framed representation, then the map β_ρ takes values in $\mathrm{Flag}(\mathbb{R}^m)$ and is a positive map in the following sense : for any finite cyclic subset $(x_1, \ldots, x_k) \in \mathcal{F}_\infty(\Sigma)$, the configuration of flags $(\beta(x_1), \ldots, \beta(x_n))$ is positive.

Proposition 5.5. *We have an identification between $\mathfrak{X}^+_{\mathrm{PGL}_m(\mathbb{C})}(\Sigma)$, and the set of conjugacy classes of (ρ, β) where ρ is a representation and $\beta : \mathcal{F}_\infty(\Sigma) \to \mathrm{Flag}(\mathbb{R}^m)$ is a positive $(\pi_1(s), \rho)$-equivariant map.*

The identification allows us to prove the following theorem

Theorem 5.6. *Positive representations are discrete.*

Proof. Let ρ be a positive representation. Let $(a, b, c) \in \mathcal{F}_\infty(\Sigma)$ be an ideal triangle of the triangulation of Σ. The flag $\beta(b)$ belongs to the set

$$\mathcal{D}_- = \{F \in \mathrm{Flag}(\mathbb{R}^m) | (\beta(a), F, \beta(c)) \text{ is a positive configuration of flags}\}$$

This set is open and disjoint from

$$\mathcal{D}_+ = \{F \in \mathrm{Flag}(\mathbb{R}^m) | (\beta(a), \beta(c), F) \text{ is a positive configuration of flags}\}$$

Hence, there exist an open neighborhood $O \subset \mathrm{PSL}_m(\mathbb{R})$ such that for all $g \in O$, we have $g \cdot \beta(b) \notin \mathcal{D}_+$.

Now let $\gamma \in \pi_1(\Sigma)$ and suppose that the image $b' = \gamma \cdot b$ is contained in the segment $(a, c) \in \mathcal{S}^1$. As β_ρ is a positive map, the quadruple of flags $(\beta(a), \beta(b), \beta(c), \rho(\gamma)\beta(b))$ is positive. We deduce easily that $\rho(\gamma) \notin O$. □

5.4. **Generalization to closed surfaces.** We can use the Farey set construction to define positive representations in the context of closed surface. Note that $\mathcal{F}_\infty(\Sigma)$ is empty for a closed surface. Hence, we need a larger object which contain $\mathcal{F}_\infty(\Sigma)$ when the surface has boundary and which is non-empty for a closed surface.

Let Σ be a compact surface (with or without boundary). Choose a hyperbolic structure on Σ with geodesic boundaries, and lift all geodesics on Σ to the universal cover. The universal cover can be identified with the hyperbolic plane with geodesic half discs removed. There is a correspondence between $\mathcal{F}_\infty(\Sigma)$ and the set of those removed geodesic half discs. We can also consider the set of endpoints of the pre images of non-boundary geodesics, and denote it by $\mathcal{G}'_\infty(\Sigma)$. And we can now define

$$\mathcal{G}_\infty(\Sigma) = \mathcal{F}_\infty(\Sigma) \cup \mathcal{G}'_\infty(\Sigma)$$

This set has the structure of a cyclic $\pi_1(\Sigma)$-set and a topology induced from $\partial\mathbb{H}$ independent of the choice of hyperbolic structure.

When the surface has boundaries and ρ is a framed positive representation, we get the positive $(\pi_1(\Sigma), \rho)$-equivariant map $\beta_\rho : \mathcal{F}_\infty(\Sigma) \to \mathrm{Flag}(\mathbb{R}^m)$. In the case of surfaces with boundary the set $\mathcal{F}_\infty(\Sigma)$ is dense in $\mathcal{G}_\infty(\Sigma)$, and thus the map can be extended uniquely to a $(\pi_1(\Sigma), \rho)$-equivariant positive continuous map :

$$\Psi_\rho : \mathcal{G}_\infty(\Sigma) \to \mathrm{Flag}(\mathbb{R}^m)$$

So for surfaces with boundaries, the following definition is equivalent to the characterization of Proposition 5.5 .

Definition 5.7. *Let Σ be a compact surface with or without boundary. A representation $\rho : \pi_1(\Sigma) \to \mathrm{PSL}_m(\mathbb{R})$ is positive, if there exists a positive ρ-equivariant map $\Psi_\rho : \mathcal{G}_\infty(\Sigma) \to \mathrm{Flag}(\mathbb{R}^m)$.*

One can then extend theorems 5.6 and 5.2 to the case of closed surfaces.

Theorem 5.8. *Let Σ be a closed surface. A positive representation $\rho : \pi_1(\Sigma) \to \mathrm{PSL}_m(\mathbb{R})$ is faithful, discrete and the image of any non-trivial element is positive.*

Moreover, the set of positive representations coincides with the Hitchin component in the representation space of $\pi_1(\Sigma)$ into $\mathrm{PSL}_m(\mathbb{R})$.

We can also identify positive representations in this context with Anosov representations as defined by Labourie [4] in terms of convex Frenet curves $\partial_\infty(\pi_1(\Sigma)) \to \mathbb{R}P^{m-1}$.

Indeed, for a positive representation ρ, the map $\Psi_\rho : \mathcal{G}_\infty(\Sigma) \to \mathrm{Flag}(\mathbb{R}^m)$ can be extended uniquely into a continuous map

$$\overline{\Psi_\rho} : \partial_\infty(\pi_1(\Sigma)) \to \mathrm{Flag}(\mathbb{R}^m)$$

that can be restricted to a ρ-equivariant curve $\partial_\infty(\pi_1(\Sigma)) \to \mathbb{R}P^{m-1}$. This curve has the desired properties to prove that the representation is Anosov.

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